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On a Ditzian–Totik Theorem*

XIE-HUA SUN

Division of Mathematics, China Institute of Metrology, Hangzhou, Zhejiang 310034, People's Republic of China

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We give the first example that a classical operator, the Bernstein interpolation process, satisfies the Ditzian–Totik theorem. © 1994 Academic Press, Inc.

1. Ditzian and Totik [1] introduced a new modulus

$$\omega_\varphi^r(f, t) := \sup_{|h| \leqslant t} \{ \|\Delta_{h\varphi}^r f\|_{C[-1,1]};\ \varphi(x) = \sqrt{1-x^2} \},$$

where

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^r (-1)^k \binom{r}{k} f(x + (k - \frac{1}{2}r)h), & \text{if } |x \pm \frac{1}{2}rh| \leqslant 1, \\ 0, & \text{otherwise,} \end{cases}$$

and proved that for any $f \in C[-1, 1]$, there exists a sequence of polynomials of degree n such that

$$\|f(x) - P_n(x)\|_{C[-1,1]} \leqslant c(r)\, \omega_\varphi^r(f, n^{-1}).$$

Recently, Ditzian and Jiang [2] further proved that for any $f \in C[-1, 1]$ and $0 \leqslant \lambda \leqslant 1$, there exists a sequence of polynomials of degree n satisfying

$$|f(x) - P_n(x)| \leqslant c(r, \lambda)\, \omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda}), \tag{1.1}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$ and

$$\omega_{\varphi^\lambda}^r(f, t) := \sup_{|h| \leqslant t} \{ \|\Delta_{h\varphi^\lambda}^r f\|_{C[-1,1]};\ \varphi(x) = \sqrt{1-x^2} \}.$$

2. In this paper we give the first example that a classical operator, the Bernstein interpolation process, satisfied the estimate (1.1). The so-called Bernstein interpolation process is defined by

$$B_n(f, x) = \sum_{k=1}^n f(x_k)\, m_k(x),$$

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where

$$m_1(x)=\tfrac{1}{4}(3l_1(x)+l_2(x)), \qquad m_n(x)=\tfrac{1}{4}(l_{n-1}(x)+3l_n(x)),$$
$$m_k(x)=\tfrac{1}{4}(l_{k-1}(x)+2l_k(x)+l_{k+1}(x)), \qquad k=2, ..., n-1,$$

and

$$l_k(x)=(-1)^{k+1}(1-x_k^2)\,T_n(x)/(n(x-x_k)), \qquad k=1, 2, ..., n,$$

are the fundamental polynomials of Lagrange interpolation at zeros of Chebyshev polynomial $T_n(x)$.

3. Our main result is the following

THEOREM. *If $f\in C[-1, 1]$ and $0\leqslant\lambda\leqslant 1$, then we have*

$$|B_n(f, x)-f(x)|\leqslant c(\lambda)\,\omega_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}),$$

where the constant $c(\lambda)$ is independent of n, f, and x, but dependent on λ.

4. We need the following

LEMMA 1 [3]. *The following estimate is valid:*

$$\sum_{k=1}^{n} m_k(x)=O(1).$$

LEMMA 2 [3]. *The following estimates are valid:*

$$\begin{aligned} m_1(x)&=O(1)\,n^{-4}\,|\cos n\theta|/|(\cos\theta-\cos\theta_1)(\cos\theta-\cos\theta_2)|,\\ m_k(x)&=O(1)\,n^{-3}\,|\cos n\theta|/|\sin\tfrac{1}{2}(\theta-\theta_{k-1})\sin\tfrac{1}{2}(\theta-\theta_k)\\ &\qquad\times\sin\tfrac{1}{2}(\theta-\theta_{k+1})|, \qquad k=2, ..., n-1,\\ m_n(x)&=O(1)\,n^{-4}\,|\cos n\theta|/|(\cos\theta-\cos\theta_{n-1})(\cos\theta-\cos\theta_n)|, \end{aligned} \tag{4.1}$$

where $x=\cos\theta$ $(0\leqslant\theta\leqslant\pi)$, $x_k=\cos\theta_k=\cos(2k-1)\pi/(2n)$.

LEMMA 3. *For $0\leqslant\lambda\leqslant 1$, the following estimate is valid*

$$\sum_{k=1}^{n} |n\sin\tfrac{1}{2}(\theta-\theta_k)|^{1-\lambda}\,|m_k(x)|=O(1). \tag{4.2}$$

Proof. Equation (4.2) follows from Lemma 1, Lemma 2, and the estimate

$$|\sin \tfrac{1}{2}(\theta-\theta_k)| \sim |\theta-\theta_k| \sim |\theta_j-\theta_k| \sim |j-k|/n, \qquad k \neq j,$$

where $|\theta_j-\theta| := \min_{1\leqslant k\leqslant n} |\theta_k-\theta|$.

LEMMA 4. *For any polynomial $p_{n-1}(x)$ of degree $\leqslant n-1$,*

$$B_n(p_{n-1}, x) = \frac{1}{4}\left[p_{n-1}\left(\cos\left(\theta+\frac{\pi}{n}\right)\right) + 2p_{n-1}(\cos\theta) + p_{n-1}\left(\cos\left(\theta-\frac{\pi}{n}\right)\right)\right] \tag{4.3}$$

holds.

Proof. Equation (4.3) follows from the invariability of the Lagrange interpolation for polynomials of degree $\leqslant n-1$.

5. *Proof of Theorem.* Let $P_{n-1}(x)$ be the polynomial of degree $\leqslant n-1$ satisfying (1.1). Since $B_n(1, x) \equiv 1$, we have

$$B_n(f, x) - f(x) = B_n(f - P_{n-1}, x) + [B_n(P_{n-1}, x) - f(x)] := I_1 + I_2. \tag{5.1}$$

Using (1.1), (4.1), (4.2), and the property of the modulus

$$\omega_{\varphi^\lambda}(f, \mu t) \leqslant c(1+\mu)\, \omega_{\varphi^\lambda}(f, t), \qquad \mu > 0,$$

and observing that

$$\frac{\delta(x_k)}{\delta(x)} = \frac{n^{-1} + \sin\theta + |\sin(\theta-\theta_k)|}{n^{-1} + \sin\theta} \leqslant 2\left(1 + \left|n \sin\frac{1}{2}(\theta-\theta_k)\right|\right),$$

we obtain

$$\begin{aligned}
|I_1| &\leqslant c(\lambda) \sum_{k=1}^{n} \omega_{\varphi^\lambda}(f, n^{-1}(\delta_n(x_k))^{1-\lambda})\, |m_k(x)| \\
&\leqslant c(\lambda)\, \omega_{\varphi^\lambda}(f, (\delta_n(x))^{1-\lambda}) \sum_{k=1}^{n} (1 + \delta_n(x_k)/\delta_n(x))^{1-\lambda})\, |m_k(x)| \\
&\leqslant c(\lambda)\, \omega_{\varphi^\lambda}(f, (\delta_n(x))^{1-\lambda}) \sum_{k=1}^{n} (1 + (n\, |\sin \tfrac{1}{2}(\theta-\theta_k)|^{1-\lambda})\, |m_k(x)| \\
&\leqslant c(\lambda)\, \omega_{\varphi^\lambda}(f, n^{-1}((\delta_n(x))^{1-\lambda}).
\end{aligned} \tag{5.2}$$

For I_2, from (4.3) it follows that

$$I_2=\frac{1}{4}\left[P_{n-1}\left(\cos\left(\theta+\frac{\pi}{n}\right)\right)-f\left(\cos\left(\theta+\frac{\pi}{n}\right)\right)\right]$$
$$+\frac{1}{4}\left[P_{n-1}\left(\cos\left(\theta-\frac{\pi}{n}\right)-f\left(\cos\left(\theta-\frac{\pi}{n}\right)\right|\right.\right.$$
$$+\frac{1}{2}\left[P_{n-1}(\cos\theta)-f(\cos\theta)\right]+\frac{1}{4}\left[f\left(\cos\left(\theta+\frac{\pi}{n}\right)\right)-f(\cos\theta)\right]$$
$$+\frac{1}{4}\left[f\left(\cos\left(\theta-\frac{\pi}{n}\right)\right)-f(\cos\theta)\right]:=I_{21}+I_{22}+I_{23}+I_{24}+I_{25}. \quad (5.3)$$

Since

$$\delta_n\left(\cos\left(\theta\pm\frac{\pi}{n}\right)\right)=n^{-1}+\left|\sin\left(\theta\pm\frac{\pi}{n}\right)\right|\leqslant c(n^{-1}+\sin\theta)=c\delta_n(x),$$

then using (1.1), we obtain

$$I_{21}+I_{22}+I_{23}=O(1)\,\omega_{\varphi^\lambda}(f,n^{-1}(\delta_n(x))^{1-\lambda}). \quad (5.4)$$

Observing that

$$1-\left[\frac{1}{2}\left(\cos\left(\theta+\frac{\pi}{n}\right)+\cos\theta\right)\right]^2\geqslant\sin^2\left(\theta+\frac{\pi}{2n}\right)$$

and according to the definition of the modulus, we have

$$\left|f\left(\cos\left(\theta+\frac{\pi}{n}\right)\right)-f(\cos\theta)\right|$$
$$\leqslant\omega_{\varphi^\lambda}\left(f,\frac{|\cos(\theta+\pi/n)-\cos\theta|}{(1-(\frac{1}{2}(\cos(\theta+\pi/n)+\cos\theta))^2)^{\lambda/2}}\right)$$
$$\leqslant c\omega_{\varphi^\lambda}\left(f,2\left|\sin\left(\theta+\frac{\pi}{2n}\right)\right|^{1-\lambda}\sin\frac{\pi}{2n}\right)$$
$$\leqslant c\omega_{\varphi^\lambda}(f,n^{-1}\delta_n(x))^{1-\lambda}. \quad (5.5)$$

Similarly,

$$\left|f\left(\cos\left(\theta-\frac{\pi}{n}\right)\right)-f\left(\cos\theta\right)\right|\leqslant c\omega_{\varphi^\lambda}(f,n^{-1}(\delta_n(x))^{1-\lambda}). \quad (5.6)$$

Combining (5.3)–(5.6) yields

$$I_2 \leqslant c\omega_{\varphi^\lambda}(f, n^{-1}(\delta_n(x))^{1-\lambda}). \tag{5.7}$$

Finally, (3.1) follows from (5.1)–(5.2) and (5.7). Q.E.D.

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